

# New determinantal representations of the $W$ -weighted Drazin inverse over the quaternion skew field.

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## Abstract

Within the framework of the theory of the column and row determinants, we obtain new determinantal representations of the  $W$ -weighted Drazin inverse over the quaternion skew field. We give determinantal representations of the  $W$ -weighted Drazin inverse by using previously introduced determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the  $W$ -weighted Drazin inverse in some special case.

## 1 Introduction

Let  $\mathbb{R}$  and  $\mathbb{C}$  be the real and complex number fields, respectively. Throughout the paper, we denote the set of all  $m \times n$  matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by  $\mathbb{H}^{m \times n}$ , and by  $\mathbb{H}_r^{m \times n}$  the set of all  $m \times n$  matrices over  $\mathbb{H}$  with a rank  $r$ . Let  $M(n, \mathbb{H})$  be the ring of  $n \times n$  quaternion matrices and  $\mathbf{I}$  be the identity matrix with the appropriate size. For  $\mathbf{A} \in \mathbb{H}^{n \times m}$ , we denote by  $\mathbf{A}^*$ ,  $\text{rank } \mathbf{A}$  the conjugate transpose (Hermitian adjoint) matrix and the rank of  $\mathbf{A}$ . The matrix  $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$  is Hermitian if  $\mathbf{A}^* = \mathbf{A}$ .

The definitions of the generalized inverse matrices may be extended to quaternion matrices.

The Moore-Penrose inverse of  $\mathbf{A} \in \mathbb{H}^{m \times n}$ , denoted by  $\mathbf{A}^\dagger$ , is the unique matrix  $\mathbf{X} \in \mathbb{H}^{n \times m}$  satisfying the following equations,

$$\mathbf{A}\mathbf{X}\mathbf{A} = \mathbf{A}; \tag{1}$$

$$\mathbf{X}\mathbf{A}\mathbf{X} = \mathbf{X}; \tag{2}$$

$$(\mathbf{A}\mathbf{X})^* = \mathbf{A}\mathbf{X}; \tag{3}$$

$$(\mathbf{X}\mathbf{A})^* = \mathbf{X}\mathbf{A}. \tag{4}$$

For  $\mathbf{A} \in \mathbb{H}^{n \times n}$  with  $k = \text{Ind } \mathbf{A}$  the smallest positive number such that  $\text{rank } \mathbf{A}^{k+1} = \text{rank } \mathbf{A}^k$  the Drazin inverse of  $\mathbf{A}$ , denoted by  $\mathbf{A}^D$ , is defined to be the unique matrix  $\mathbf{X}$  that satisfying (1.2) and the following equations,

$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{A}; \tag{5}$$

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k. \tag{6}$$

In particular, when  $Ind\mathbf{A} = 1$ , then the matrix  $\mathbf{X}$  is called the group inverse and is denoted by  $\mathbf{X} = \mathbf{A}^g$ .

If  $Ind\mathbf{A} = 0$ , then  $\mathbf{A}$  is nonsingular, and  $\mathbf{A}^D \equiv \mathbf{A}^\dagger = \mathbf{A}^{-1}$ .

Cline and Greville [1] extended the Drazin inverse of square matrix to rectangular matrix, which can be generalized to the quaternion algebra as follows. For  $\mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\mathbf{W} \in \mathbb{H}^{n \times m}$ , the  $\mathbf{W}$ -weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$ , denoted by  $\mathbf{A}_{d,W}$ , is the unique solution to equations,

$$(\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^k; \quad (7)$$

$$\mathbf{XWAWX} = \mathbf{X}; \quad (8)$$

$$\mathbf{AWX} = \mathbf{XWA}, \quad (9)$$

where  $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ . It is denoted by  $\mathbf{X} = \mathbf{A}_{d,\mathbf{W}}$ . The properties of the complex  $\mathbf{W}$ -weighted Drazin inverse can be found in [1–6]. These properties can be generalized to  $\mathbb{H}$ . If  $\mathbf{A} \in \mathbb{H}^{m \times n}$  with respect to  $\mathbf{W} \in \mathbb{H}^{n \times m}$  and  $k = \max\{Ind(\mathbf{AW}), Ind(\mathbf{WA})\}$ , then

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A}((\mathbf{WA})^D)^2 = ((\mathbf{AW})^D)^2 \mathbf{A}, \quad (10)$$

$$\mathbf{A}_{d,\mathbf{W}}\mathbf{W} = (\mathbf{WA})^D, \mathbf{WA}_{d,\mathbf{W}} = (\mathbf{AW})^D. \quad (11)$$

The problem of determinantal representation of generalized inverse matrices only recently begun to be decided through the theory of the column-row determinants introduced in [7, 8]. The theory of row and column determinants develops the classical approach to a definition of a determinant, as alternating sum of products of the entries of matrix but with a predetermined order of factors in each terms of the determinant. A determinant of a quadratic matrix with noncommutative elements is often called the noncommutative determinant. Unlike other known noncommutative determinants such as determinants of Dieudonné [9], Study [10], Moore [11, 12], Chen [13], quasideterminants of Gelfand-Retakh [14], the double determinant built on the theory of the column-row determinants has properties similar to a usual determinant, in particular it can be expand along arbitrary rows and columns. This property is necessary for determinantal representations of an inverse and generalized inverse matrices. Determinantal representations of the Moore-Penrose inverse and the Drazin inverse over the quaternion skew-field have been obtained in [15, 16] and [17], respectively. Determinantal representations of an outer inverse  $\mathbf{A}_{T,S}^{(2)}$  is introduced in [18, 19] using the column-row determinants as well. Recall that an outer inverse of a matrix  $\mathbf{A}$  over complex field with prescribed range space  $T$  and null space  $S$  is a solution of (1.2) with restrictions,

$$\mathcal{R}(\mathbf{X}) = T, \mathcal{N}(\mathbf{X}) = S.$$

Within the framework of the theory of the column-row determinants Song [20] also gave a determinantal representation  $\mathbf{W}$ -weighted Drazin inverse over the quaternion skew-field using a characterization of the  $\mathbf{W}$ -weighted Drazin inverse by an outer inverse  $\mathbf{A}_{T,S}^{(2)}$ . But in obtaining of this determinantal representation

is used auxiliary matrices which different from  $\mathbf{A}$  or its powers. In this paper we obtain determinantal representations of the  $W$ -weighted Drazin inverse of  $\mathbf{A} \in \mathbb{H}^{m \times n}$  with respect to  $\mathbf{W} \in \mathbb{H}^{n \times m}$  by using only their entries.

The paper is organized as follows. We start with some basic concepts and results from the theory of the row and column determinants and give the determinantal representations of the inverse, the Moore-Penrose inverse, and the Drazin inverse over the quaternion skew field in Section 2. In Section 3, we obtain determinantal representations of the  $W$ -weighted Drazin inverse by using introduced above determinantal representations of the Drazin inverse, the Moore-Penrose inverse, and the limit representations of the  $W$ -weighted Drazin inverse in some special case. In Section 4, we show a numerical example to illustrate the main result.

## 2 Elements of the theory of the column and row determinants

For a quadratic matrix  $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$  we define  $n$  row determinants and  $n$  column determinants as follows. Suppose  $S_n$  is the symmetric group on the set  $I_n = \{1, \dots, n\}$ .

**Definition 2.1** *The  $i$ th row determinant of  $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$  is defined for all  $i = \overline{1, n}$  by putting*

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}},$$

$$\sigma = (i i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$

with conditions  $i_{k_2} < i_{k_3} < \dots < i_{k_r}$  and  $i_{k_t} < i_{k_t+s}$  for  $t = \overline{2, r}$  and  $s = \overline{1, l_t}$ .

**Definition 2.2** *The  $j$ th column determinant of  $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$  is defined for all  $j = \overline{1, n}$  by putting*

$$\text{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} i_{k_r}} \dots a_{j j_{k_1+l_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j},$$

$$\tau = (j_{k_r+l_r} \dots j_{k_r+1} j_{k_r}) \dots (j_{k_2+l_2} \dots j_{k_2+1} j_{k_2}) (j_{k_1+l_1} \dots j_{k_1+1} j_{k_1} j),$$

with conditions,  $j_{k_2} < j_{k_3} < \dots < j_{k_r}$  and  $j_{k_t} < j_{k_t+s}$  for  $t = \overline{2, r}$  and  $s = \overline{1, l_t}$ .

Suppose  $\mathbf{A}^{ij}$  denotes the submatrix of  $\mathbf{A}$  obtained by deleting both the  $i$ th row and the  $j$ th column. Let  $\mathbf{a}_{\cdot j}$  be the  $j$ th column and  $\mathbf{a}_{i \cdot}$  be the  $i$ th row of  $\mathbf{A}$ . Suppose  $\mathbf{A}_{\cdot j}(\mathbf{b})$  denotes the matrix obtained from  $\mathbf{A}$  by replacing its  $j$ th column with the column  $\mathbf{b}$ , and  $\mathbf{A}_{i \cdot}(\mathbf{b})$  denotes the matrix obtained from  $\mathbf{A}$  by replacing its  $i$ th row with the row  $\mathbf{b}$ .

The following theorem has a key value in the theory of the column and row determinants.

**Theorem 2.1** [7] If  $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$  is Hermitian, then  $\text{rdet}_1 \mathbf{A} = \dots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \dots = \text{cdet}_n \mathbf{A} \in \mathbb{R}$ .

Since all column and row determinants of a Hermitian matrix over  $\mathbb{H}$  are equal, we can define the determinant of a Hermitian matrix  $\mathbf{A} \in M(n, \mathbb{H})$ . By definition, we put  $\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}$ , for all  $i = \overline{1, n}$ .

The determinant of a Hermitian matrix has properties similar to a usual determinant. They are completely explored in [7, 8] by its row and column determinants. They can be summarized by the following theorems.

**Theorem 2.2** If the  $i$ th row of a Hermitian matrix  $\mathbf{A} \in M(n, \mathbb{H})$  is replaced with a left linear combination of its other rows, i.e.  $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}$ , where  $c_l \in \mathbb{H}$  for all  $l = \overline{1, k}$  and  $\{i, i_l\} \subset I_n$ , then

$$\text{rdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}) = \text{cdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \dots + c_k \mathbf{a}_{i_k.}) = 0.$$

**Theorem 2.3** If the  $j$ th column of a Hermitian matrix  $\mathbf{A} \in M(n, \mathbb{H})$  is replaced with a right linear combination of its other columns, i.e.  $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k$ , where  $c_l \in \mathbb{H}$  for all  $l = \overline{1, k}$  and  $\{j, j_l\} \subset J_n$ , then

$$\text{cdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) = \text{rdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \dots + \mathbf{a}_{.j_k} c_k) = 0.$$

The determinant of a Hermitian matrix also has a property of expansion along arbitrary rows and columns using row and column determinants of submatrices. So, we were able to get determinantal representations of an inverse and generalized inverse matrices as follows.

**Theorem 2.4** [7] If for a Hermitian matrix  $\mathbf{A} \in M(n, \mathbb{H})$ ,

$$\det \mathbf{A} \neq 0,$$

then there exist a unique right inverse matrix  $(R\mathbf{A})^{-1}$  and a unique left inverse matrix  $(L\mathbf{A})^{-1}$  of a nonsingular  $\mathbf{A}$ , where  $(R\mathbf{A})^{-1} = (L\mathbf{A})^{-1} =: \mathbf{A}^{-1}$ , and the right and left inverse matrices possess the following determinantal representations

$$(R\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} R_{11} & R_{21} & \dots & R_{n1} \\ R_{12} & R_{22} & \dots & R_{n2} \\ \dots & \dots & \dots & \dots \\ R_{1n} & R_{2n} & \dots & R_{nn} \end{pmatrix}, \quad (12)$$

$$(L\mathbf{A})^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{n1} \\ L_{12} & L_{22} & \dots & L_{n2} \\ \dots & \dots & \dots & \dots \\ L_{1n} & L_{2n} & \dots & L_{nn} \end{pmatrix}, \quad (13)$$

where  $\det \mathbf{A} = \sum_{j=1}^n a_{ij} \cdot R_{ij} = \sum_{i=1}^n L_{ij} \cdot a_{ij}$ ,

$$R_{ij} = \begin{cases} -\text{rdet}_j \mathbf{A}_{.j}^{ii}(\mathbf{a}_{.i}), & i \neq j, \\ \text{rdet}_k \mathbf{A}^{ii}, & i = j, \end{cases} \quad L_{ij} = \begin{cases} -\text{cdet}_i \mathbf{A}_{i.}^{jj}(\mathbf{a}_{j.}), & i \neq j, \\ \text{cdet}_k \mathbf{A}^{jj}, & i = j, \end{cases}$$

and  $\mathbf{A}_{.j}^{ii}(\mathbf{a}_{.i})$  is obtained from  $\mathbf{A}$  by both replacing the  $j$ th column with the  $i$ th column and deleting the  $i$ th row and column,  $\mathbf{A}_{i.}^{jj}(\mathbf{a}_{j.})$  is obtained by both replacing the  $i$ th row with the  $j$ th row and deleting the  $j$ th row and column, respectively,  $I_n = \{1, \dots, n\}$ ,  $k = \min\{I_n \setminus \{i\}\}$  for all  $i, j = \overline{1, n}$ .

We shall use the following notations. Let  $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$  and  $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$  be subsets of the order  $1 \leq k \leq \min\{m, n\}$ . By  $\mathbf{A}_{\beta}^{\alpha}$  denote the submatrix of  $\mathbf{A}$  determined by the rows indexed by  $\alpha$  and the columns indexed by  $\beta$ . Then  $\mathbf{A}_{\alpha}^{\alpha}$  denotes the principal submatrix determined by the rows and columns indexed by  $\alpha$ . If  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian, then by  $|\mathbf{A}_{\alpha}^{\alpha}|$  denote the corresponding principal minor of  $\det \mathbf{A}$ . For  $1 \leq k \leq n$ , the collection of strictly increasing sequences of  $k$  integers chosen from  $\{1, \dots, n\}$  is denoted by  $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 \leq \dots \leq \alpha_k \leq n\}$ . For fixed  $i \in \alpha$  and  $j \in \beta$ , let  $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$ ,  $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$ .

Denote by  $\mathbf{a}_{.j}^*$  and  $\mathbf{a}_{i.}^*$  the  $j$ th column and the  $i$ th row of  $\mathbf{A}^*$  and by  $\mathbf{a}_{.j}^{(m)}$  and  $\mathbf{a}_{i.}^{(m)}$  the  $j$ th column and the  $i$ th row of  $\mathbf{A}^m$ , respectively.

The following theorem give determinantal representations of the Moore-Penrose inverse over the quaternion skew field  $\mathbb{H}$ .

**Theorem 2.5** [15] *If  $\mathbf{A} \in \mathbb{H}_r^{m \times n}$ , then the Moore-Penrose inverse  $\mathbf{A}^+ = (a_{ij}^+) \in \mathbb{H}^{n \times m}$  possess the following determinantal representations:*

$$a_{ij}^+ = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^* \mathbf{A})_{.i}(\mathbf{a}_{.j}^*))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^* \mathbf{A})_{\beta}^{\beta}|}, \quad (14)$$

or

$$a_{ij}^+ = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A} \mathbf{A}^*)_{j.}(\mathbf{a}_{i.}^*))_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A} \mathbf{A}^*)_{\alpha}^{\alpha}|}. \quad (15)$$

for all  $i = \overline{1, n}$ ,  $j = \overline{1, m}$ .

**Proposition 2.1** [21] *If  $\text{Ind}(\mathbf{A}) = k$ , then  $\mathbf{A}^D = \mathbf{A}^k(\mathbf{A}^{2k+1})^+ \mathbf{A}^k$ .*

Using the determinantal representations of the Moore-Penrose inverse (14) and (15), and Proposition 2.1 we have obtained the following determinantal representations of the Drazin inverse for an arbitrary square matrix over  $\mathbb{H}$ . Denote by  $\hat{\mathbf{a}}_s$  and  $\check{\mathbf{a}}_t$  the  $s$ th column of  $(\mathbf{A}^{2k+1})^* \mathbf{A}^k =: \hat{\mathbf{A}} = (\hat{a}_{ij}) \in \mathbb{H}^{n \times n}$  and the  $t$ th row of  $\mathbf{A}^k (\mathbf{A}^{2k+1})^* =: \check{\mathbf{A}} = (\check{a}_{ij}) \in \mathbb{H}^{n \times n}$ , respectively, for all  $s, t = \overline{1, n}$ .

**Theorem 2.6** [17] *If  $\mathbf{A} \in M(n, \mathbb{H})$  with  $\text{Ind} \mathbf{A} = k$  and  $\text{rank} \mathbf{A}^{k+1} = \text{rank} \mathbf{A}^k = r$ , then the Drazin inverse  $\mathbf{A}^D$  possess the determinantal representations*

$$a_{ij}^D = \frac{\sum_{t=1}^n a_{it}^{(k)} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t((\mathbf{A}^{2k+1})^* (\mathbf{A}^{2k+1})_{.t}(\hat{\mathbf{a}}_{.j}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^{2k+1})^* (\mathbf{A}^{2k+1})_{\beta}^{\beta}|} \quad (16)$$

and

$$a_{ij}^D = \frac{\sum_{s=1}^n \left( \sum_{\alpha \in I_{r,n}\{s\}} \text{rdet}_s \left( \left( \mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^* \right)_{\cdot s} (\tilde{\mathbf{a}}_i) \right)_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum_{\alpha \in I_{r,n}} \left| \left( \mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^* \right)_{\alpha}^{\alpha} \right|} \quad (17)$$

In the special case, when  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian, we can obtain simpler determinantal representations of the Drazin inverse.

**Theorem 2.7** [17] *If  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian with  $\text{Ind } \mathbf{A} = k$  and  $\text{rank } \mathbf{A}^{k+1} = \text{rank } \mathbf{A}^k = r$ , then the Drazin inverse  $\mathbf{A}^D = (a_{ij}^D) \in \mathbb{H}^{n \times n}$  possess the following determinantal representations:*

$$a_{ij}^D = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left( \left( \mathbf{A}^{k+1} \right)_{\cdot i} (\mathbf{a}_{\cdot j}^k) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left( \mathbf{A}^{k+1} \right)_{\beta}^{\beta} \right|}, \quad (18)$$

or

$$a_{ij}^D = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left( \left( \mathbf{A}^{k+1} \right)_{j \cdot} (\mathbf{a}_i^{(k)}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,n}} \left| \left( \mathbf{A}^{k+1} \right)_{\alpha}^{\alpha} \right|}. \quad (19)$$

Note that **the determinantal rank** of  $\mathbf{A} \in \mathbb{C}^{m \times n}$  can be obtained as the largest order of a non-zero principle minor in the Hermitian matrices  $\mathbf{A}^* \mathbf{A}$  or  $\mathbf{A} \mathbf{A}^*$ .

We shall also need the following facts about the eigenvalues of quaternion matrices. Due to the noncommutativity of quaternions, there are two types of eigenvalues. A quaternion  $\lambda$  is said to be a right eigenvalue of  $\mathbf{A} \in M(n, \mathbb{H})$  if  $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda$  for some nonzero quaternion column-vector  $\mathbf{x}$  with quaternion components. Similarly  $\lambda$  is a left eigenvalue if  $\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}$  for some nonzero quaternion column-vector  $\mathbf{x}$  with quaternion components. The theory on the left eigenvalues of quaternion matrices has been investigated in particular in [22–24]. The theory on the right eigenvalues of quaternion matrices is more developed. In particular we note [25–30].

**Proposition 2.2** [29] *Let  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian. Then  $\mathbf{A}$  has exactly  $n$  real right eigenvalues.*

Right and left eigenvalues are in general unrelated [31], but it is not for Hermitian matrices. Suppose  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian and  $\lambda \in \mathbb{R}$  is its right eigenvalue, then  $\mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \lambda = \lambda \cdot \mathbf{x}$ . This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues,  $\lambda \in \mathbb{R}$ , the matrix  $\lambda \mathbf{I} - \mathbf{A}$  is Hermitian.

**Definition 2.3** *If  $t \in \mathbb{R}$ , then for a Hermitian matrix  $\mathbf{A} \in M(n, \mathbb{H})$  the polynomial  $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$  is said to be the characteristic polynomial of  $\mathbf{A}$ .*

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well. We can prove the following theorem by analogy to the commutative case (see, e.g. [32]).

**Theorem 2.8** *If  $\mathbf{A} \in M(n, \mathbb{H})$  is Hermitian, then  $p_{\mathbf{A}}(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \dots + (-1)^n d_n$ , where  $d_k$  is the sum of principle minors of  $\mathbf{A}$  of order  $rk$ ,  $1 \leq k < n$ , and  $d_n = \det \mathbf{A}$ .*

### 3 Determinantal representations of the W-weighted Drazin inverse for an arbitrary matrix

Determinantal representations W-weighted Drazin inverse of complex matrices have been received by full-rank factorization in [33] and by a limit representation in [34]. For an arbitrary matrix over the field of complex numbers,  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , we denote by  $\mathcal{R}(\mathbf{A})$  the range of  $\mathbf{A}$  and by  $\mathcal{N}(\mathbf{A})$  the null space of  $\mathbf{A}$ . For an arbitrary matrix over the quaternion skew field,  $\mathbf{A} \in \mathbb{H}^{m \times n}$ , we denote by

$$\begin{aligned}\mathcal{R}_r(\mathbf{A}) &= \{\mathbf{y} \in \mathbb{H}^m : \mathbf{y} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{H}^n\}, \text{ the column right space of } \mathbf{A}, \\ \mathcal{N}_r(\mathbf{A}) &= \{\mathbf{y} \in \mathbb{H}^n : \mathbf{A}\mathbf{x} = 0\}, \text{ the right null space of } \mathbf{A}, \\ \mathcal{R}_l(\mathbf{A}) &= \{\mathbf{y} \in \mathbb{H}^n : \mathbf{y} = \mathbf{x}\mathbf{A}, \mathbf{x} \in \mathbb{H}^m\}, \text{ the column left space of } \mathbf{A}, \\ \mathcal{N}_l(\mathbf{A}) &= \{\mathbf{y} \in \mathbb{H}^m : \mathbf{x}\mathbf{A} = 0\}, \text{ the left null space of } \mathbf{A}.\end{aligned}$$

Through the theory of the column-row determinants, a determinantal representation W-weighted Drazin inverse over the quaternion skew-field for the first time has been obtained in [20] by the following theorem.

**Theorem 3.1** *Let  $\mathbf{A} \in \mathbb{H}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{H}^{n \times m}$  with  $k = \max\{Ind(\mathbf{A}\mathbf{W}), Ind(\mathbf{W}\mathbf{A})\}$  and  $\text{rank}(\mathbf{A}\mathbf{W})^k = s$ . Suppose that  $\mathbf{B} \in \mathbb{H}_{n-s}^{n \times (n-s)}$  and  $\mathbf{C}^* \in \mathbb{H}_{m-s}^{m \times (m-s)}$  are of full ranks and*

$$\begin{aligned}\mathcal{R}_r(\mathbf{B}) &= \mathcal{N}_r((\mathbf{W}\mathbf{A})^k), \quad \mathcal{N}_r(\mathbf{C}) = \mathcal{R}_r((\mathbf{A}\mathbf{W})^k), \\ \mathcal{R}_l(\mathbf{C}) &= \mathcal{N}_l((\mathbf{A}\mathbf{W})^k), \quad \mathcal{N}_l(\mathbf{B}) = \mathcal{R}_l((\mathbf{W}\mathbf{A})^k).\end{aligned}$$

Denote

$$\mathbf{M} = \begin{bmatrix} \mathbf{W}\mathbf{A}\mathbf{W} & \mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}.$$

Then the W-weighted Drazin inverse  $\mathbf{A}_{d,\mathbf{W}} = (a)_{ij} \in \mathbb{H}^{n \times m}$  has the following determinantal representations:

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} L_{ki} m_{kj}^*}{\det \mathbf{M}^* \mathbf{M}}, \quad i = \overline{1, m}, j = \overline{1, n}, \quad (20)$$

or

$$a_{ij} = \frac{\sum_{k=1}^{m+n-s} m_{ik}^* R_{jk}}{\det \mathbf{M} \mathbf{M}^*}, \quad i = \overline{1, m}, j = \overline{1, n}, \quad (21)$$

where  $L_{ij}$  are the left  $(ij)$ -th cofactor of  $\mathbf{M}^* \mathbf{M}$  and  $R_{ij}$  are the right  $(ij)$ -th cofactor of  $\mathbf{M} \mathbf{M}^*$ , respectively, for all  $i, j = \overline{1, m+n-s}$ .

As can be seen, the auxiliary matrices  $\mathbf{B}$  and  $\mathbf{C}$  have been used in the determinantal representations (20) and (21). In this paper we escape it. Below we give determinantal representations of the  $\mathbf{W}$ -weighted Drazin inverse of an arbitrary matrix  $\mathbf{A} \in \mathbb{H}^{m \times n}$  with respect to the matrix  $\mathbf{W} \in \mathbb{H}^{n \times m}$  by using the determinantal representations of the Drazin inverse, of the Moore-Penrose inverse, and the limit representation of the  $\mathbf{W}$ -weighted Drazin inverse in some particular case.

### 3.1 Determinantal representations of the $\mathbf{W}$ -weighted Drazin inverse by using determinantal representations of the Drazin inverse

Denote  $\mathbf{WA} =: \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$  and  $\mathbf{AW} =: \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$ . Due to Theorem 2.6, we denote an entry of the Drazin inverse  $\mathbf{U}^D$  by

$$u_{ij}^{D,1} = \frac{\sum_{t=1}^n u_{it}^{(k)} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left( ((\mathbf{U}^{2k+1})^* (\mathbf{U}^{2k+1})_{.t} (\hat{\mathbf{u}}_{.j}))_{\beta}^{\beta} \right)}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{U}^{2k+1})^* (\mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \quad (22)$$

or

$$u_{ij}^{D,2} = \frac{\sum_{s=1}^n \left( \sum_{\alpha \in I_{r,n}\{s\}} \text{rdet}_s \left( ((\mathbf{U}^{2k+1} (\mathbf{U}^{2k+1})^*)_{.s} (\check{\mathbf{u}}_{i.}))_{\alpha}^{\alpha} \right) \right) u_{sj}^{(k)}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{U}^{2k+1} (\mathbf{U}^{2k+1})^*)_{\alpha}^{\alpha} \right|} \quad (23)$$

where  $\hat{\mathbf{u}}_{.s}$  and  $\check{\mathbf{u}}_{t.}$  the  $s$ th column of  $(\mathbf{U}^{2k+1})^* \mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n \times n}$  and the  $t$ th row of  $\mathbf{U}^k (\mathbf{U}^{2k+1})^* =: \check{\mathbf{U}} = (\check{u}_{ij}) \in \mathbb{H}^{n \times n}$ , respectively for all  $s, t = \overline{1, n}$ ,  $r = \text{rank } \mathbf{U}^{k+1} = \text{rank } \mathbf{U}^k$ . Then by (10) we can obtain the following determinantal representations of  $\mathbf{A}_{d,\mathbf{W}} = (a_{ij}^{d,\mathbf{W}}) \in \mathbb{H}^{m \times n}$ ,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^n a_{iq} (u_{qj}^D)^{(2)} \quad (24)$$

where  $(u_{qj}^D)^{(2)} = \sum_{p=1}^n u_{qp}^{D,l} u_{pj}^{D,f}$  for all  $l, f = \overline{1, 2}$ , and  $u_{ij}^{D,1}$  from (22) and  $u_{ij}^{D,2}$  from (23). Similarly using  $\mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$  we have the following determinantal representations of  $\mathbf{A}_{d,\mathbf{W}}$ ,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^m (v_{iq}^D)^{(2)} a_{qj}. \quad (25)$$



The first factor is one of the four possible equations  $(v_{iq}^D)^{(2)} = \sum_{p=1}^m v_{ip}^{D,l} v_{pq}^{D,f}$  for all  $l, f = \overline{1, 2}$ , and an entry of the Drazin inverse  $\mathbf{V}^D$  is denoting by

$$v_{ij}^{D,1} = \frac{\sum_{t=1}^m v_{it}^{(k)} \sum_{\beta \in J_{r,m}\{t\}} \text{cdet}_t \left( (\mathbf{V}^{2k+1})^* (\mathbf{V}^{2k+1})_{.t} (\hat{\mathbf{v}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{V}^{2k+1})^* (\mathbf{V}^{2k+1})_{\beta}^{\beta} \right|} \quad (26)$$

or

$$v_{ij}^{D,2} = \frac{\sum_{s=1}^m \left( \sum_{\alpha \in I_{r,m}\{s\}} \text{rdet}_s \left( (\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{.s} (\check{\mathbf{v}}_{i.}) \right)_{\alpha}^{\alpha} \right) v_{sj}^{(k)}}{\sum_{\alpha \in I_{r,m}} \left| (\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha} \right|}, \quad (27)$$

where  $\hat{\mathbf{v}}_{.s}$  and  $\check{\mathbf{v}}_{.t}$  the  $s$ th column of  $(\mathbf{V}^{2k+1})^* \mathbf{V}^k =: \hat{\mathbf{V}} = (\hat{v}_{ij}) \in \mathbb{H}^{m \times m}$  and the  $t$ th row of  $\mathbf{V}^k (\mathbf{V}^{2k+1})^* =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m \times m}$ , respectively for all  $s, t = \overline{1, m}$ ,  $r = \text{rank } \mathbf{V}^{k+1} = \text{rank } \mathbf{V}^k$ .

### 3.2 Determinantal representations of the $\mathbf{W}$ -weighted Drazin inverse by using determinantal representations of the Moore-Penrose inverse

Consider the general algebraic structures (GAS) of the matrices  $\mathbf{A} \in \mathbb{H}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{H}^{n \times m}$ ,  $\mathbf{A}^+ \in \mathbb{H}^{n \times m}$ ,  $\mathbf{W}^+ \in \mathbb{H}^{m \times n}$  and  $\mathbf{A}_{d,\mathbf{W}} \in \mathbb{H}^{m \times n}$  with  $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\}$  (e.g., [2–5]). Let exist  $\mathbf{L} \in \mathbb{H}^{m \times m}$  and  $\mathbf{Q} \in \mathbb{H}^{n \times n}$  such that

$$\mathbf{A} = \mathbf{L} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \mathbf{Q}^{-1}, \quad \mathbf{W} = \mathbf{Q} \begin{bmatrix} \mathbf{W}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22} \end{bmatrix} \mathbf{L}^{-1}.$$

Then

$$\mathbf{A}^+ = \mathbf{Q} \begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{L}^{-1}, \quad \mathbf{W}^+ = \mathbf{L} \begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1},$$

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{L} \begin{bmatrix} (\mathbf{W}_{11} \mathbf{A}_{11} \mathbf{W}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1},$$

where  $\mathbf{L}, \mathbf{Q}, \mathbf{A}_{11}, \mathbf{W}_{11}$  are non-singular matrices, and  $\mathbf{A}_{22} \mathbf{W}_{22}, \mathbf{W}_{22} \mathbf{A}_{22}$  are nilpotent matrices. The follow theorem due to [5] can be expanded to  $\mathbb{H}$ .

**Theorem 3.2** *Let  $\mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\mathbf{W} \in \mathbb{H}^{n \times m}$  such that  $\mathbf{A}_{22} \mathbf{W}_{22}$  and  $\mathbf{W}_{22} \mathbf{A}_{22}$  are nilpotent matrices of index  $k$  in GAS form. Then the weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  can be written as matrix expression involving the Moore-Penrose inverse,*

$$\mathbf{A}_{d,\mathbf{W}} = \left\{ (\mathbf{A}\mathbf{W})^k [(\mathbf{A}\mathbf{W})^{2k+1}]^+ (\mathbf{A}\mathbf{W})^k \right\} \mathbf{W}^+, \quad (28)$$

where  $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\}$ .

Similarly can be obtained the following theorem.

**Theorem 3.3** *Let  $\mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\mathbf{W} \in \mathbb{H}^{n \times m}$  such that  $\mathbf{A}_{22}\mathbf{W}_{22}$  and  $\mathbf{W}_{22}\mathbf{A}_{22}$  are nilpotent matrices of index  $k$  in GAS form. Then the  $W$ -weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  can be written as the following matrix expression,*

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{W}^+ \left\{ (\mathbf{W}\mathbf{A})^k [(\mathbf{W}\mathbf{A})^{2k+1}]^+ (\mathbf{W}\mathbf{A})^k \right\}, \quad (29)$$

where  $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\}$ .

**Proof.** Since  $\mathbf{W}_{22}\mathbf{A}_{22}$  is a nilpotent matrix of index  $k$ , then due to the GAS of  $\mathbf{A}$ ,  $\mathbf{W}$  and their generalized inverses we have the following Jordan canonical forms,

$$\begin{aligned} \mathbf{W}\mathbf{A} &= \mathbf{Q} \begin{bmatrix} \mathbf{W}_{11}\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{22}\mathbf{A}_{22} \end{bmatrix} \mathbf{Q}^{-1}, \quad (\mathbf{W}\mathbf{A})^k = \mathbf{Q} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1}, \\ [(\mathbf{W}\mathbf{A})^{2k+1}]^+ &= \mathbf{Q} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1}. \end{aligned}$$

Simple computing of  $\mathbf{W}^+ \left\{ (\mathbf{W}\mathbf{A})^k [(\mathbf{W}\mathbf{A})^{2k+1}]^+ (\mathbf{W}\mathbf{A})^k \right\}$  proves the theorem,

$$\begin{aligned} &\mathbf{W}^+ \left\{ (\mathbf{W}\mathbf{A})^k [(\mathbf{W}\mathbf{A})^{2k+1}]^+ (\mathbf{W}\mathbf{A})^k \right\} = \\ &\mathbf{L} \begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11})^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1} = \\ &\mathbf{L} \begin{bmatrix} \mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^k(\mathbf{W}_{11}\mathbf{A}_{11})^{-2k-1}(\mathbf{W}_{11}\mathbf{A}_{11})^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1} = \\ &\mathbf{L} \begin{bmatrix} \mathbf{W}_{11}^{-1}(\mathbf{W}_{11}\mathbf{A}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1} = \\ &\mathbf{L} \begin{bmatrix} (\mathbf{W}_{11}\mathbf{A}_{11}\mathbf{W}_{11})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1} = \mathbf{A}_{d,\mathbf{W}}. \blacksquare \end{aligned}$$

Using (28), an entry  $a_{ij}^{d,\mathbf{W}}$  of the  $W$ -weighted Drazin inverse  $\mathbf{A}_{d,\mathbf{W}}$  can be obtained as follows

$$a_{ij}^{d,\mathbf{W}} = \sum_{s=1}^m \sum_{t=1}^m \sum_{l=1}^m v_{is}^{(k)} \left( v_{st}^{(2k+1)} \right)^+ v_{tl}^{(k)} w_{lj}^+ \quad (30)$$

for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ . Denote by  $\check{\mathbf{w}}_t$ , the  $t$ th row of  $\mathbf{V}^k \mathbf{W}^* =: \check{\mathbf{W}} = (\check{w}_{ij}) \in \mathbb{H}^{m \times n}$  for all  $t = \overline{1, m}$ . It follows from  $\sum_l v_{tl}^{(k)} \mathbf{w}_l^* = \check{\mathbf{w}}_t$ , and (15) that

$$\sum_{l=1}^m v_{tl}^{(k)} w_{lj}^+ = \sum_{l=1}^m v_{tl}^{(k)} \cdot \frac{\sum_{\alpha \in I_{r_1, n} \{j\}} \text{rdet}_j (\mathbf{W} \mathbf{W}^*)_{j \cdot} (\mathbf{w}_l^*)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r_1, n}} |(\mathbf{W} \mathbf{W}^*)_{\alpha}^{\alpha}|} = \frac{\sum_{\alpha \in I_{r_1, n} \{j\}} \text{rdet}_j \left( (\mathbf{W} \mathbf{W}^*)_{j \cdot} (\check{\mathbf{w}}_t)_{\alpha}^{\alpha} \right)}{\sum_{\alpha \in I_{r_1, n}} |(\mathbf{W} \mathbf{W}^*)_{\alpha}^{\alpha}|}, \quad (31)$$

where  $r_1 = \text{rank } \mathbf{W}$ . Similarly, denote by  $\check{\mathbf{v}}_i$ , the  $t$ th row of  $\mathbf{V}^k (\mathbf{V}^{2k+1})^* =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m \times m}$  for all  $t = \overline{1, m}$ . It follows from  $\sum_s v_{is}^{(k)} (\mathbf{v}_s^{(2k+1)})^* = \check{\mathbf{v}}_i$ , and (15) that

$$\sum_{s=1}^m v_{is}^{(k)} \left( v_{st}^{(2k+1)} \right)^+ = \sum_{s=1}^m v_{is}^{(k)} \cdot \frac{\sum_{\alpha \in I_{r, m} \{t\}} \text{rdet}_t \left( \left( \mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^* \right)_t (\mathbf{v}_s^{(2k+1)})^* \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r, m}} |(\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha}|} = \frac{\sum_{\alpha \in I_{r, m} \{t\}} \text{rdet}_t \left( \left( \mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^* \right)_t (\check{\mathbf{v}}_i)_{\alpha}^{\alpha} \right)}{\sum_{\alpha \in I_{r, m}} |(\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha}|}, \quad (32)$$

where  $r = \text{rank } \mathbf{W}^{k+1} = \text{rank } \mathbf{W}^k$ . Using (31) and (32) in (30) we obtain the following determinantal representation of  $\mathbf{A}_{d, \mathbf{W}}$ ,

$$a_{ij}^{d, \mathbf{W}} = \frac{\sum_{t=1}^m \sum_{\alpha \in I_{r, m} \{t\}} \text{rdet}_t \left( \left( \mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^* \right)_t (\check{\mathbf{v}}_i)_{\alpha}^{\alpha} \right) \sum_{\alpha \in I_{r_1, n} \{j\}} \text{rdet}_j \left( (\mathbf{W} \mathbf{W}^*)_{j \cdot} (\check{\mathbf{w}}_t)_{\alpha}^{\alpha} \right)}{\sum_{\alpha \in I_{r, m}} |(\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha}| \sum_{\alpha \in I_{r_1, n}} |(\mathbf{W} \mathbf{W}^*)_{\alpha}^{\alpha}|} \quad (33)$$

Thus we have proved the following theorem.

**Theorem 3.4** *Let  $\mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\mathbf{W} \in \mathbb{H}_{r_1}^{n \times m}$  with  $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\}$  and  $r = \text{rank}(\mathbf{A}\mathbf{W})^{k+1} = \text{rank}(\mathbf{W}\mathbf{A})^k$ . Then the  $\mathbf{W}$ -weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  possesses the determinantal representation (33), where  $\mathbf{V} = \mathbf{A}\mathbf{W}$ ,  $\check{\mathbf{V}} = \mathbf{V}^k (\mathbf{V}^{2k+1})^*$ , and  $\check{\mathbf{W}} = \mathbf{V}^k \mathbf{W}^*$ .*

Similarly we have the following theorem.

**Theorem 3.5** Let  $\mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\mathbf{W} \in \mathbb{H}_{r_1}^{n \times m}$  with  $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$  and  $r = \text{rank}(\mathbf{WA})^{k+1} = \text{rank}(\mathbf{WA})^k$ . Then the  $W$ -weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  possesses the following determinantal representation,

$$a_{ij}^{d, \mathbf{W}} = \frac{\sum_{t=1}^n \sum_{\beta \in J_{r_1, m}\{i\}} \text{cdet}_i((\mathbf{W}^* \mathbf{W})_{.i}(\hat{\mathbf{w}}_{.t}))_{\beta}^{\beta} \sum_{\beta \in J_{r, n}\{t\}} \text{cdet}_t\left(\left((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1}\right)_{.t}(\hat{\mathbf{u}}_{.j})\right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1, m}} \left|(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}\right| \sum_{\beta \in J_{r, n}} \left|((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{\beta}^{\beta}\right|} \quad (34)$$

where  $\mathbf{U} = \mathbf{WA}$ ,  $\hat{\mathbf{U}} = (\mathbf{U}^{2k+1})^* \mathbf{U}^k$ , and  $\hat{\mathbf{W}} = \mathbf{W}^* \mathbf{U}^k$ .

**Proof.** Using (29), an entry  $a_{ij}^{d, \mathbf{W}}$  of the  $W$ -weighted Drazin inverse  $\mathbf{A}_{d, \mathbf{W}}$  can be obtained as follows

$$a_{ij}^{d, \mathbf{W}} = \sum_{s=1}^n \sum_{t=1}^n \sum_{l=1}^n w_{is}^+ u_{st}^{(k)} \left(u_{tl}^{(2k+1)}\right)^+ u_{lj}^{(k)} \quad (35)$$

for all  $i = \overline{1, m}$ ,  $j = \overline{1, n}$ . Denote by  $\hat{\mathbf{w}}_{.t}$  the  $t$ th column of  $\mathbf{W}^* \mathbf{U}^k =: \hat{\mathbf{W}} = (\hat{w}_{ij}) \in \mathbb{H}^{m \times n}$  for all  $t = \overline{1, n}$ . It follows from  $\sum_t \mathbf{w}_{.s}^* u_{st}^{(k)} = \hat{\mathbf{w}}_{.t}$  and (14) that

$$\sum_{s=1}^n w_{is}^+ u_{st}^{(k)} = \sum_{s=1}^n \frac{\sum_{\beta \in J_{r_1, m}\{i\}} \text{cdet}_i(\mathbf{W}^* \mathbf{W})_{.i}(\mathbf{w}_{.s}^*)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1, m}} \left|(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}\right|} \cdot u_{st}^{(k)} = \frac{\sum_{\beta \in J_{r_1, m}\{i\}} \text{cdet}_i((\mathbf{W}^* \mathbf{W})_{.i}(\hat{\mathbf{w}}_{.t}))_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1, m}} \left|(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}\right|}, \quad (36)$$

where  $r_1 = \text{rank } \mathbf{W}$ . Similarly, denote by  $\hat{\mathbf{u}}_{.j}$  the  $j$ th column of  $(\mathbf{U}^{2k+1})^* \mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n \times n}$  for all  $j = \overline{1, n}$ . It follows from  $\sum_l \left(\mathbf{u}_{.l}^{(2k+1)}\right)^* u_{lj}^{(k)} = \hat{\mathbf{u}}_{.j}$  and

(14) that

$$\begin{aligned} \sum_{l=1}^n \left( u_{tl}^{(2k+1)} \right)^+ u_{lj}^{(k)} = \\ \sum_{l=1}^n \frac{\sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left( \left( (\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{.t} \left( \mathbf{u}_{.l}^{(2k+1)} \right)^* \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left( (\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{\beta}^{\beta} \right|} \cdot u_{lj}^{(k)} = \\ \frac{\sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left( \left( (\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{.t} (\hat{\mathbf{u}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| \left( (\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{\beta}^{\beta} \right|}, \quad (37) \end{aligned}$$

where  $r = \text{rank}(\mathbf{AW})^{k+1} = \text{rank}(\mathbf{AW})^k$ . Using the equations (37) and (36) in (35), we obtain (34). ■

### 3.3 Determinantal representations of the W-weighted Drazin inverse in some special case

In this subsection we consider the determinantal representation of the W-weighted Drazin inverse of  $\mathbf{A} \in \mathbb{H}^{m \times n}$  with respect to  $\mathbf{W} \in \mathbb{H}^{n \times m}$  in a special case, when  $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$  and  $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$  are Hermitian. Then for the determinantal representations their Drazin inverse we can use (18) and (19). For Hermitian matrix, we apply the method, which consists of theorem on the limit representation of the Drazin inverse, lemmas on rank of matrices and on characteristic polynomial. This method was used at first in [35] and then in [15, 34, 36, 37]. By analogy to the complex case [38] we have the following limit representations of the W-weighted Drazin inverse,

$$\mathbf{A}_{d,W} = \lim_{\lambda \rightarrow 0} (\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})^{-1} (\mathbf{AW})^k \mathbf{A} \quad (38)$$

and

$$\mathbf{A}_{d,W} = \lim_{\lambda \rightarrow 0} \mathbf{A} (\mathbf{WA})^k (\lambda \mathbf{I}_n + (\mathbf{WA})^{k+2})^{-1} \quad (39)$$

where  $\lambda \in \mathbb{R}_+$ , and  $\mathbb{R}_+$  is a set of the real positive numbers.

Denote by  $\mathbf{v}_{.j}^{(k)}$  and  $\mathbf{v}_{i.}^{(k)}$  the  $j$ th column and the  $i$ th row of  $\mathbf{V}^k$  respectively. Denote by  $\bar{\mathbf{V}}^k := (\mathbf{AW})^k \mathbf{A} \in \mathbb{H}^{m \times n}$  and  $\bar{\mathbf{W}} = \mathbf{WAW} \in \mathbb{H}^{n \times m}$ .

**Lemma 3.1** *If  $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$  with  $\text{Ind} \mathbf{V} = k$ , then*

$$\text{rank} \left( \mathbf{V}^{k+2} \right)_{.i} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \leq \text{rank} \left( \mathbf{V}^{k+2} \right). \quad (40)$$

**Proof.** We have  $\mathbf{V}^{k+2} = \bar{\mathbf{V}}^k \bar{\mathbf{W}}$ . Let  $\mathbf{P}_{is}(-\bar{w}_{js}) \in \mathbb{H}^{m \times m}$ , ( $s \neq i$ ), be a matrix with  $-\bar{w}_{js}$  in the  $(i, s)$  entry, 1 in all diagonal entries, and 0 in others. The elementary matrix  $\mathbf{P}_{is}(-\bar{w}_{js})$ , ( $s \neq i$ ), is a matrix of an elementary

transformation. It follows that

$$(\mathbf{V}^{k+2})_{\cdot i} \left( \bar{\mathbf{v}}_{\cdot j}^{(k)} \right) \cdot \prod_{s \neq i} \mathbf{P}_{is}(-\bar{w}_{js}) = \begin{pmatrix} \sum_{s \neq j} \bar{v}_{1s}^{(k)} \bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum_{s \neq j} \bar{v}_{1s}^{(k)} \bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{s \neq j} \bar{v}_{ms}^{(k)} \bar{w}_{s1} & \dots & \bar{v}_{mj}^{(k)} & \dots & \sum_{s \neq j} \bar{v}_{ms}^{(k)} \bar{w}_{sm} \end{pmatrix}.$$

$i-th$

We have the next factorization of the obtained matrix.

$$\begin{pmatrix} \sum_{s \neq j} \bar{v}_{1s}^{(k)} \bar{w}_{s1} & \dots & \bar{v}_{1j}^{(k)} & \dots & \sum_{s \neq j} \bar{v}_{1s}^{(k)} \bar{w}_{sm} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{s \neq j} \bar{v}_{ms}^{(k)} \bar{w}_{s1} & \dots & \bar{v}_{mj}^{(k)} & \dots & \sum_{s \neq j} \bar{v}_{ms}^{(k)} \bar{w}_{sm} \end{pmatrix} =$$

$i-th$

$$= \begin{pmatrix} \bar{v}_{11}^{(k)} & \bar{v}_{12}^{(k)} & \dots & \bar{v}_{1n}^{(k)} \\ \bar{v}_{21}^{(k)} & \bar{v}_{22}^{(k)} & \dots & \bar{v}_{2n}^{(k)} \\ \dots & \dots & \dots & \dots \\ \bar{v}_{m1}^{(k)} & \bar{v}_{m2}^{(k)} & \dots & \bar{v}_{mn}^{(k)} \end{pmatrix} \begin{pmatrix} \bar{w}_{11} & \dots & 0 & \dots & \bar{w}_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{w}_{n1} & \dots & 0 & \dots & \bar{w}_{nm} \end{pmatrix} j-th.$$

$i-th$

Denote  $\tilde{\mathbf{W}} := \begin{pmatrix} \bar{w}_{11} & \dots & 0 & \dots & \bar{w}_{1m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \bar{w}_{n1} & \dots & 0 & \dots & \bar{w}_{nm} \end{pmatrix} j-th$ . The matrix  $\tilde{\mathbf{W}}$  is obtained

from  $\bar{\mathbf{W}} = \mathbf{W}\mathbf{A}\mathbf{W}$  by replacing all entries of the  $j$ th row and the  $i$ th column with zeroes except for 1 in the  $(i, j)$  entry. Since elementary transformations of a matrix do not change a rank, then  $\text{rank } \mathbf{V}_{\cdot i}^{k+2} \left( \bar{\mathbf{v}}_{\cdot j}^{(k)} \right) \leq \min \left\{ \text{rank } \bar{\mathbf{V}}^k, \text{rank } \tilde{\mathbf{W}} \right\}$ . It is obvious that

$$\begin{aligned} \text{rank } \bar{\mathbf{V}}^k &= \text{rank } (\mathbf{A}\mathbf{W})^k \mathbf{A} \geq \text{rank } (\mathbf{A}\mathbf{W})^{k+2}, \\ \text{rank } \tilde{\mathbf{W}} &\geq \text{rank } \mathbf{W}\mathbf{A}\mathbf{W} \geq \text{rank } (\mathbf{A}\mathbf{W})^{k+2}. \end{aligned}$$

From this the inequality (40) follows immediately. ■

The next lemma is proved similarly.

**Lemma 3.2** *If  $\mathbf{W}\mathbf{A} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$  with  $\text{Ind } \mathbf{U} = k$ , then*

$$\text{rank } (\mathbf{U}^{k+2})_{i \cdot} \left( \bar{\mathbf{u}}_j^{(k)} \right) \leq \text{rank } (\mathbf{U}^{k+2}),$$

where  $\bar{\mathbf{U}}^k := \mathbf{A}(\mathbf{W}\mathbf{A})^k \in \mathbb{H}^{m \times n}$

Analogs of the characteristic polynomial are considered in the following two lemmas.

**Lemma 3.3** If  $\mathbf{A}\mathbf{W} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$  is Hermitian with  $\text{Ind}\mathbf{V} = k$  and  $t \in \mathbb{R}$ , then

$$\text{cdet}_i(\lambda \mathbf{I}_m + \mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.j}^{(k)}) = c_1^{(ij)} \lambda^{n-1} + c_2^{(ij)} \lambda^{n-2} + \dots + c_n^{(ij)}, \quad (41)$$

where  $c_n^{(ij)} = \text{cdet}_i(\mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.j}^{(k)})$  and  $c_s^{(ij)} = \sum_{\beta \in J_{s,n}\{i\}} \text{cdet}_i((\mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.j}^{(k)}))_{\beta}^{\beta}$  for all  $s = \overline{1, n-1}$ ,  $i, j = \overline{1, n}$ .

**Proof.** Consider the Hermitian matrix  $(t\mathbf{I} + \mathbf{V}^{k+2})_{.i}(\mathbf{v}_{.i}^{(k+2)}) \in \mathbb{H}^{n \times n}$ . Taking into account Theorem 2.8 we obtain

$$\det(\lambda \mathbf{I} + \mathbf{V}^{k+2})_{.i}(\mathbf{v}_{.i}^{(k+2)}) = d_1 \lambda^{n-1} + d_2 \lambda^{n-2} + \dots + d_n, \quad (42)$$

where  $d_s = \sum_{\beta \in J_{s,n}\{i\}} |(\mathbf{V}^{k+2})_{\beta}^{\beta}|$  is the sum of all principal minors of order  $s$  that contain the  $i$ -th column for all  $s = \overline{1, n-1}$  and  $d_n = \det(\mathbf{V}^{k+2})$ . Consequently

$$\text{we have } \mathbf{v}_{.i}^{(k+2)} = \begin{pmatrix} \sum_l \bar{v}_{1l}^{(k)} \bar{w}_{li} \\ \sum_l \bar{v}_{2l}^{(k)} \bar{w}_{li} \\ \vdots \\ \sum_l \bar{v}_{nl}^{(k)} \bar{w}_{li} \end{pmatrix} = \sum_l \bar{\mathbf{v}}_{.l}^{(k)} \bar{w}_{li}, \text{ where } \bar{\mathbf{v}}_{.l}^{(k)} \text{ is the } l\text{th column-}$$

vector of  $\bar{\mathbf{V}}^k = (\mathbf{A}\mathbf{W})^k \mathbf{A}$  and  $\mathbf{W}\mathbf{A}\mathbf{W} = \bar{\mathbf{W}} = (\bar{w}_{li})$  for all  $l = \overline{1, n}$ . Taking into account Theorem 2.1, we obtain on the one hand

$$\begin{aligned} \det(\lambda \mathbf{I} + \mathbf{V}^{k+2})_{.i}(\mathbf{v}_{.i}^{(k+2)}) &= \text{cdet}_i(\lambda \mathbf{I} + \mathbf{V}^{k+2})_{.i}(\mathbf{v}_{.i}^{(k+2)}) = \\ &= \sum_l \text{cdet}_i(\lambda \mathbf{I} + \mathbf{V}^{k+2})_{.l}(\bar{\mathbf{v}}_{.l}^{(k)} \bar{w}_{li}) = \sum_l \text{cdet}_i(\lambda \mathbf{I} + \mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.l}^{(k)}) \cdot \bar{w}_{li} \end{aligned} \quad (43)$$

On the other hand having changed the order of summation, we get for all  $s = \overline{1, n-1}$

$$\begin{aligned} d_s &= \sum_{\beta \in J_{s,n}\{i\}} \det(\mathbf{V}^{k+2})_{\beta}^{\beta} = \sum_{\beta \in J_{s,n}\{i\}} \text{cdet}_i(\mathbf{V}^{k+2})_{\beta}^{\beta} = \\ &= \sum_{\beta \in J_{s,n}\{i\}} \sum_l \text{cdet}_i((\mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.l}^{(k)} \bar{w}_{li}))_{\beta}^{\beta} = \\ &= \sum_l \sum_{\beta \in J_{s,n}\{i\}} \text{cdet}_i((\mathbf{V}^{k+2})_{.i}(\bar{\mathbf{v}}_{.l}^{(k)}))_{\beta}^{\beta} \cdot \bar{w}_{li}. \end{aligned} \quad (44)$$

By substituting (43) and (44) in (42), and equating factors at  $\bar{w}_{li}$  when  $l = j$ , we obtain the equality (41). ■

By analogy can be proved the following lemma.

**Lemma 3.4** If  $\mathbf{W}\mathbf{A} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$  is Hermitian with  $\text{Ind}\mathbf{U} = k$  and  $t \in \mathbb{R}$ , then

$$\text{rdet}_j(\lambda \mathbf{I} + \mathbf{U}^{k+2})_{.j}(\bar{\mathbf{u}}_{.i}^{(k)}) = r_1^{(ij)} \lambda^{n-1} + r_2^{(ij)} \lambda^{n-2} + \dots + r_n^{(ij)},$$

where  $r_n^{(ij)} = \text{rdet}_j(\mathbf{U}^{k+2})_{j \cdot} (\bar{\mathbf{u}}_i^{(k)})$  and  $r_s^{(ij)} = \sum_{\alpha \in I_{s,n} \setminus \{j\}} \text{rdet}_j \left( (\mathbf{U}^{k+2})_{j \cdot} (\bar{\mathbf{u}}_i^{(k)}) \right)_{\alpha}^{\beta}$  for all  $s = \overline{1, n-1}$  and  $i, j = \overline{1, n}$ .

**Theorem 3.6** If  $\mathbf{A} \in \mathbb{H}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{H}^{n \times m}$ , and  $\mathbf{AW} = \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$  is Hermitian with  $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$  and  $\text{rank}(\mathbf{AW})^{k+1} = \text{rank}(\mathbf{AW})^k = r$ , then the  $W$ -weighted Drazin inverse  $\mathbf{A}_{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n}$  with respect to  $\mathbf{W}$  possess the following determinantal representations:

$$a_{ij}^{d,W} = \frac{\sum_{\beta \in J_{r,m} \setminus \{i\}} \text{cdet}_i \left( (\mathbf{AW})_{\cdot i}^{k+2} \left( \bar{\mathbf{v}}_{\cdot j}^{(k)} \right) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{AW})_{\cdot i}^{k+2} \right|_{\beta}^{\beta}}, \quad (45)$$

where  $\bar{\mathbf{v}}_{\cdot j}^{(k)}$  is the  $j$ th column of  $\bar{\mathbf{V}}^k = (\mathbf{AW})^k \mathbf{A}$  for all  $j = \overline{1, m}$ .

**Proof.** The matrix  $(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})^{-1} \in \mathbb{H}^{m \times m}$  is a full-rank Hermitian matrix. Taking into account Theorem 2.4 it has an inverse, which we represent as a left inverse matrix

$$(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})^{-1} = \frac{1}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} \begin{pmatrix} L_{11} & L_{21} & \dots & L_{m1} \\ L_{12} & L_{22} & \dots & L_{m2} \\ \dots & \dots & \dots & \dots \\ L_{1m} & L_{2m} & \dots & L_{mm} \end{pmatrix},$$

where  $L_{ij}$  is a left  $ij$ -th cofactor of a matrix  $\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}$ . Then we have

$$\begin{aligned} & (\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})^{-1} (\mathbf{AW})^k \mathbf{A} = \\ & = \frac{1}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} \begin{pmatrix} \sum_{s=1}^m L_{s1} \bar{v}_{s1}^{(k)} & \sum_{s=1}^m L_{s1} \bar{v}_{s2}^{(k)} & \dots & \sum_{s=1}^m L_{s1} \bar{v}_{sn}^{(k)} \\ \sum_{s=1}^m L_{s2} \bar{v}_{s1}^{(k)} & \sum_{s=1}^m L_{s2} \bar{v}_{s2}^{(k)} & \dots & \sum_{s=1}^m L_{s2} \bar{v}_{sn}^{(k)} \\ \dots & \dots & \dots & \dots \\ \sum_{s=1}^m L_{sm} \bar{v}_{s1}^{(k)} & \sum_{s=1}^m L_{sm} \bar{v}_{s2}^{(k)} & \dots & \sum_{s=1}^m L_{sm} \bar{v}_{sn}^{(k)} \end{pmatrix}. \end{aligned}$$

By (38) and using the definition of a left cofactor, we obtain

$$\mathbf{A}_{d,W} = \lim_{\alpha \rightarrow 0} \begin{pmatrix} \frac{\text{cdet}_1(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})_{\cdot 1}(\bar{\mathbf{v}}_{\cdot 1}^{(k)})}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} & \dots & \frac{\text{cdet}_1(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})_{\cdot 1}(\bar{\mathbf{v}}_{\cdot n}^{(k)})}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} \\ \dots & \dots & \dots \\ \frac{\text{cdet}_n(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})_{\cdot n}(\bar{\mathbf{v}}_{\cdot 1}^{(k)})}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} & \dots & \frac{\text{cdet}_n(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})_{\cdot n}(\bar{\mathbf{v}}_{\cdot n}^{(k)})}{\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2})} \end{pmatrix}. \quad (46)$$

By Theorem 2.8 we have

$$\det(\lambda \mathbf{I}_m + (\mathbf{AW})^{k+2}) = \lambda^m + d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \dots + d_m,$$



where  $d_s = \sum_{\beta \in J_{s,m}} \left| \left( \lambda \mathbf{I}_m + (\mathbf{A}\mathbf{W})^{k+2} \right)_{\beta}^{\beta} \right|$  is a sum of principal minors of  $(\mathbf{A}\mathbf{W})^{k+2}$  of order  $s$  for all  $s = \overline{1, m-1}$  and  $d_m = \det(\mathbf{A}\mathbf{W})^{k+2}$ . Since  $\text{rank}(\mathbf{A}\mathbf{W})^{k+2} = \text{rank}(\mathbf{A}\mathbf{W})^{k+1} = \text{rank}(\mathbf{A}\mathbf{W})^k = r$ , then  $d_m = d_{m-1} = \dots = d_{r+1} = 0$ . It follows that  $\det(\lambda \mathbf{I}_m + (\mathbf{A}\mathbf{W})^{k+2}) = \lambda^m + d_1 \lambda^{m-1} + d_2 \lambda^{m-2} + \dots + d_r \lambda^{m-r}$ . Using (41) we have

$$\text{cdet}_i(\lambda \mathbf{I}_m + (\mathbf{A}\mathbf{W})^{k+2})_{.i} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) = c_1^{(ij)} \lambda^{m-1} + c_2^{(ij)} \lambda^{m-2} + \dots + c_m^{(ij)}$$

for  $i = \overline{1, m}$  and  $j = \overline{1, n}$ , where  $c_s^{(ij)} = \sum_{\beta \in J_{s,m}\{i\}} \text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$  for all  $s = \overline{1, m-1}$  and  $c_m^{(ij)} = \text{cdet}_i(\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right)$ . We shall prove that  $c_k^{(ij)} = 0$ , when  $k \geq r+1$  for  $i = \overline{1, m}$  and  $j = \overline{1, n}$ . By Lemma 3.1  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right) \leq r$ , then the matrix  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)$  has no more  $r$  right-linearly independent columns. Consider  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$ , when  $\beta \in J_{s,m}\{i\}$ . It is a principal submatrix of  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)$  of order  $s \geq r+1$ . Deleting both its  $i$ -th row and column, we obtain a principal submatrix of order  $s-1$  of  $(\mathbf{A}\mathbf{W})^{k+2}$ . We denote it by  $\mathbf{M}$ . The following cases are possible.

- Let  $s = r+1$  and  $\det \mathbf{M} \neq 0$ . In this case all columns of  $\mathbf{M}$  are right-linearly independent. The addition of all of them on one coordinate to columns of  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$  keeps their right-linear independence. Hence, they are basis in a matrix  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$ , and the  $i$ -th column is the right linear combination of its basis columns. From this by Theorem 2.3, we get  $\text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta} = 0$ , when  $\beta \in J_{s,n}\{i\}$  and  $s = r+1$ .
- If  $s = r+1$  and  $\det \mathbf{M} = 0$ , than  $p$ , ( $p < s$ ), columns are basis in  $\mathbf{M}$  and in  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$ . Then by Theorem 2.3,  $\text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta} = 0$  as well.
- If  $s > r+1$ , then  $\det \mathbf{M} = 0$  and  $p$ , ( $p < r$ ), columns are basis in the both matrices  $\mathbf{M}$  and  $\left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$ . Then by Theorem 2.3, we also have  $\text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta} = 0$ .

Thus in all cases we have  $\text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta} = 0$ , when  $\beta \in J_{s,m}\{i\}$  and  $r+1 \leq s < m$ . From here if  $r+1 \leq s < m$ , then

$$c_s^{(ij)} = \sum_{\beta \in J_{s,m}\{i\}} \text{cdet}_i \left( (\mathbf{A}\mathbf{W})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right)_{\beta}^{\beta} = 0,$$

and  $c_m^{(ij)} = \text{cdet}_i \left( (\mathbf{AW})_{.i}^{k+2} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right) = 0$  for all  $i, j = \overline{1, n}$ .

Hence,  $\text{cdet}_i \left( (\lambda \mathbf{I} + (\mathbf{AW})^{k+2})_{.i} \left( \bar{\mathbf{v}}_{.j}^{(k)} \right) \right) = c_1^{(ij)} \lambda^{m-1} + c_2^{(ij)} \lambda^{m-2} + \dots + c_r^{(ij)} \lambda^{m-r}$  for  $i = \overline{1, m}$  and  $j = \overline{1, n}$ . By substituting these values in the matrix from (46), we obtain

$$\mathbf{A}_{d,W} = \lim_{\lambda \rightarrow 0} \begin{pmatrix} \frac{c_1^{(11)} \lambda^{m-1} + \dots + c_r^{(11)} \lambda^{m-r}}{\lambda^m + d_1 \lambda^{m-1} + \dots + d_r \lambda^{m-r}} & \dots & \frac{c_1^{(1n)} \lambda^{m-1} + \dots + c_r^{(1n)} \lambda^{m-r}}{\lambda^m + d_1 \lambda^{m-1} + \dots + d_r \lambda^{m-r}} \\ \dots & \dots & \dots \\ \frac{c_1^{(m1)} \lambda^{m-1} + \dots + c_r^{(m1)} \lambda^{m-r}}{\lambda^m + d_1 \lambda^{m-1} + \dots + d_r \lambda^{m-r}} & \dots & \frac{c_1^{(mn)} \lambda^{m-1} + \dots + c_r^{(mn)} \lambda^{m-r}}{\lambda^m + d_1 \lambda^{m-1} + \dots + d_r \lambda^{m-r}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{c_r^{(11)}}{d_r} & \dots & \frac{c_r^{(1n)}}{d_r} \\ \dots & \dots & \dots \\ \frac{c_r^{(m1)}}{d_r} & \dots & \frac{c_r^{(mn)}}{d_r} \end{pmatrix}.$$

Here  $c_r^{(ij)} = \sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i \left( (\mathbf{A}^{k+1})_{.i} \left( \mathbf{a}_{.j}^{(k)} \right) \right)_{\beta}^{\beta}$  and  $d_r = \sum_{\beta \in J_{r,m}} \left| (\mathbf{A}^{k+1})_{\beta}^{\beta} \right|$ .

Thus, we have obtained the determinantal representation of  $\mathbf{A}_{d,W}$  by (45). ■

By analogy can be proved the following theorem.

**Theorem 3.7** *If  $\mathbf{A} \in \mathbb{H}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{H}^{n \times m}$ , and  $\mathbf{WA} = \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$  is Hermitian with  $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$  and  $\text{rank}(\mathbf{WA})^{k+1} = \text{rank}(\mathbf{WA})^k = r$ , then the  $W$ -weighted Drazin inverse  $\mathbf{A}_{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n}$  with respect to  $\mathbf{W}$  possess the following determinantal representations:*

$$a_{ij}^{d,W} = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left( (\mathbf{WA})_{.j}^{k+2} (\bar{\mathbf{u}}_{.i}^{(k)}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{WA})_{\alpha}^{k+2} \right|_{\alpha}}. \quad (47)$$

where  $\bar{\mathbf{u}}_{.i}^{(k)}$  is the  $i$ th row of  $\bar{\mathbf{U}}^k = \mathbf{A}(\mathbf{WA})^k$  for all  $i = \overline{1, n}$ .

## 4 An example

In this section, we give an example to illustrate our results. Let us consider the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & i & 0 \\ k & 1 & i \\ 1 & 0 & 0 \\ 1 & -k & -j \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix}.$$

Then

$$\mathbf{V} = \mathbf{AW} = \begin{pmatrix} -k & -j & 0 & i \\ -1-j & i+k & j & 1+j \\ k & 0 & i & 0 \\ -i+k & 1-j & i & i-k \end{pmatrix}, \quad \mathbf{U} = \mathbf{WA} = \begin{pmatrix} i & j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and  $\text{rank } \mathbf{W} = 3, \text{rank } \mathbf{V} = 3, \text{rank } \mathbf{V}^3 = \text{rank } \mathbf{V}^2 = 2, \text{rank } \mathbf{U}^2 = \text{rank } \mathbf{U} = 2$ . Therefore,  $\text{Ind } \mathbf{V} = 2, \text{Ind } \mathbf{U} = 1$ , and  $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\} = 2$ . It's evident that obtaining the  $W$ -weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  by using the matrix  $\mathbf{U}$  by (34) is more convenient. We have

$$\begin{aligned}\mathbf{U}^2 &= \begin{pmatrix} -1 & i+k & 0 \\ 0 & -1 & \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}^5 = \begin{pmatrix} i & 2+3j & 0 \\ 0 & k & \\ 0 & 0 & 0 \end{pmatrix}, \\ (\mathbf{U}^5)^* &= \begin{pmatrix} -i & 0 & 0 \\ 2-3j & -k & \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{U}^5)^* \mathbf{U}^5 = \begin{pmatrix} 1 & -2i-3k & 0 \\ 2i+3k & 14 & \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{\mathbf{U}} = (\mathbf{U}^5)^* \mathbf{U}^2 &= \begin{pmatrix} i & 1+j & 0 \\ -2+3j & -i+6k & \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{W}^* = \begin{pmatrix} -k & j & 0 \\ 0 & -k & 1 \\ -i & 0 & 0 \\ 0 & 1 & k \end{pmatrix}, \\ \mathbf{W}^* \mathbf{W} &= \begin{pmatrix} 2 & i & -j & j \\ -i & 2 & 0 & -2k \\ j & 0 & 1 & 0 \\ -j & 2k & 0 & 2 \end{pmatrix}, \quad \hat{\mathbf{W}} = \mathbf{W}^* \mathbf{U}^2 = \begin{pmatrix} -k & 1-2j & 0 \\ 0 & i+k & 0 \\ i & 1+j & 0 \\ 0 & -1 & 0 \end{pmatrix}.\end{aligned}$$

Since by (34)

$$\begin{aligned}a_{11}^{d, \mathbf{W}} &= \frac{\sum_{t=1}^3 \sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.t}))_{\beta}^{\beta} \sum_{\beta \in J_{2,3}\{t\}} \text{cdet}_t(((\mathbf{U}^5)^* \mathbf{U}^5)_{.t}(\hat{\mathbf{u}}_{.1}))_{\beta}^{\beta}}{\sum_{\beta \in J_{3,4}} |(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}| \sum_{\beta \in J_{2,3}} |((\mathbf{U}^5)^* \mathbf{U}^5)_{\beta}^{\beta}|},\end{aligned}$$

where

$$\begin{aligned}\sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.1}))_{\beta}^{\beta} &= \\ \text{cdet}_1 \begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} &+ \text{cdet}_1 \begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0,\end{aligned}$$

$$\begin{aligned}\sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.2}))_{\beta}^{\beta} &= -2j, \quad \sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.3}))_{\beta}^{\beta} = 0, \\ \sum_{\beta \in J_{3,4}} |(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}| &= 2,\end{aligned}$$

and

$$\sum_{\beta \in J_{2,3}\{1\}} \text{cdet}_1 \left( \left( (\mathbf{U}^5)^* \mathbf{U}^5 \right)_{.1} (\hat{\mathbf{u}}_{.1}) \right)_{\beta}^{\beta} = \text{cdet}_1 \begin{pmatrix} i & -2i-3k \\ -2+3j & 14 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = i,$$

$$\sum_{\beta \in J_{2,3}\{2\}} \text{cdet}_2 \left( \left( (\mathbf{U}^5)^* \mathbf{U}^5 \right)_{.2} (\hat{\mathbf{u}}_{.1}) \right)_{\beta}^{\beta} = 0, \\ \sum_{\beta \in J_{2,3}\{3\}} \text{cdet}_3 \left( \left( (\mathbf{U}^5)^* \mathbf{U}^5 \right)_{.3} (\hat{\mathbf{u}}_{.1}) \right)_{\beta}^{\beta} = 0, \quad \sum_{\beta \in J_{2,3}} \left| \left( (\mathbf{U}^5)^* \mathbf{U}^5 \right)_{\beta}^{\beta} \right| = 1,$$

then

$$a_{11}^{d,\mathbf{W}} = \frac{(0 \cdot i) + (-2j \cdot 0) + (0 \cdot 0)}{2 \cdot 1} = 0.$$

Continuing in the same way, we finally get,

$$\mathbf{A}_{d,\mathbf{W}} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (48)$$

By (14) we can obtain

$$(\mathbf{U}^5)^+ = \begin{pmatrix} -i & -3+2j & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{W}\mathbf{A})^D = \mathbf{U}^2 (\mathbf{U}^5)^+ \mathbf{U}^2 = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We verify (48) by (11). Indeed,

$$\mathbf{W}\mathbf{A}_{d,\mathbf{W}} = \begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ -1 & 5i-2k & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -i & -5 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\mathbf{W}\mathbf{A})^D.$$

We also obtain the W-weighted Drazin inverse of  $\mathbf{A}$  with respect to  $\mathbf{W}$  by (24), then we have

$$\mathbf{A}_{d,\mathbf{W}} = \mathbf{A} ((\mathbf{W}\mathbf{A})^D)^2 = \begin{pmatrix} 0 & -i & 0 \\ -k & 6+5i & 0 \\ -1 & 5i+5k & 0 \\ -1 & 5i+6k & 0 \end{pmatrix}, \quad (49)$$

The W-weighted Drazin inverse in (49) different from (48). It can be explained that the Jordan normal form of  $\mathbf{W}\mathbf{A}$  is unique only up to the order of the Jordan blocks. We get their complete equality, if  $\mathbf{A}_{d,\mathbf{W}}$  from (49) be left-multiply by

the nonsingular matrix  $\mathbf{P}$  which is the product of multiplication of the following elementary matrices,

$$\mathbf{P} = \mathbf{P}_{2,4}(-k) \cdot \mathbf{P}_{4,3}(-1) \cdot \mathbf{P}_{3,4}(-6) \cdot \mathbf{P}_{4,1}(-j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 7 & -6 \\ -j & 0 & -1 & 1 \end{pmatrix}.$$

Note that we used Maple with the package CLIFFORD in the calculations.

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